Detail-Preserving Fourth-Order Nonlinear PDE-Based Image Restoration Framework

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Abstract—**A novel fourth-order Partial Differential Equation (PDE) - based image restoration technique is proposed in this work. It is based on a well-posed fourth order nonlinear diffusion based model with some properly chosen boundary conditions, which is combined to a two dimensional filter kernel. An explicit iterative finite difference method based numerical approximation algorithm is then constructed for solving the PDE model. It is stable and converges fast to the solution of the differential model, which represents the recovered image. The proposed filtering approach removes successfully the additive noise, overcome the unintended effects, such as the blurring and staircasing, and preserves successfully the edges and other image details. As it results from the obtained method comparison results, this approach outperforms not only the classic 2D image filters that often generate the undesired blurring effect, but also some nonlinear second order partial differential equation based smoothing schemes that produce the blocky effect.**

Index Terms—**image denoising and restoration, fourth-order PDE model, nonlinear diffusion, additive Gaussian noise, finite difference method, numerical approximation scheme**

I. INTRODUCTION

Feature-preserving image denoising and restoration still represents a challenging image processing task. The conventional image filters image filters may generate undesirable effects, such as blurring, during the smoothing process [1], which affect the image details.

The second-order Partial Differential Equation (PDE) – based restoration methods provide much better noise removal results and overcome successfully the blurring effect. They have been widely used in this domain in the last three decades, since P. Perona and J. Malik introduced their anisotropic diffusion-based restoration scheme [2].

Variational PDE restoration models, inspired by the influential Total Variation (TV) Denoising [3], have been also developed in this period [4]-[6]. While these nonlinear second-order diffusion-based denoising models preserve the edges and other essential features of the processed image, they could also generate the unintended staircase effect.

The nonlinear fourth-order PDE restoration models inspired by the You-Kaveh isotropic diffusion scheme solve properly this drawback, removing successfully the additive noise and overcoming the staircasing [7]. Unfortunately, they may produce over-filtering and multiplicative (speckle) noise, which may destroy some details.

We have performed a high amount of research in the PDE-based image restoration field, developing some improved second and fourth order diffusion-based models that overcome the drawbacks of the existing denoising methods derived from Perona-Malik and You-Kaveh scheme [8], [9]. In this paper we propose a novel fourth-order anisotropic diffusion-based filtering technique that provides a successful additive Gaussian noise removal and overcomes both the blurring and the staircasing effects.

The proposed denoising framework is based on a nonlinear fourth-order PDE model combined to a 2D filter kernel, which is described in the next section. A finite difference-based numerical approximation scheme is constructed for this diffusion model in the third section. Then, some restoration experiments and method comparison are presented in the fourth section. The conclusions of this research work are drawn in the last section.

II. FOURTH-ORDER ANISOTROPIC DIFFUSION-BASED FILTERING MODELS

A nonlinear fourth-order anisotropic diffusion model for image restoration is proposed here. It is based on the following parabolic fourth-order PDE with boundary conditions, which is combined to a 2D filter kernel:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \varphi\left(\left\|\nabla K_\sigma * u\right\|\right)\nabla^2\left(\psi\left(\left\|\Delta u\right\|\right)\nabla^2 u\right) + \lambda\left(u - u_0\right) = 0 \\[2mm]
u\left(x, y, 0\right) = u_0\left(x, y\right), \quad \forall\left(x, y\right) \in \Omega \\[2mm]
\dfrac{\partial u}{\partial \vec{n}} = 0 \\[2mm]
u\left(t, x, y\right) = 0, \quad \forall\left(x, y\right) \in \partial\Omega
\end{cases}
\tag{1}
$$

where the image domain is $\Omega \subseteq R^2$, u_0 represents the observed image and the parameter $\lambda \in [0, 1)$.

The fourth-order diffusion component of the PDE-based model given by (1) uses the following diffusivity function:

$$\psi : [0, \infty) \rightarrow [0, \infty)$$
$$\psi(s) = \varepsilon \sqrt{\frac{|\mu(\|\nabla u\|) + t\xi|}{|\alpha s^3 + \beta|}} \quad (2)$$

where the coefficients $\beta \in [0, 5)$ and $\alpha, \varepsilon, \xi \in [0, 1)$.

This function is properly modeled for noise removal, being positive, monotonically decreasing and converging to 0 [2]. The fourth-order diffusion-based component $\nabla^2 \left(\psi(\|\Delta u\|) \nabla^2 u \right)$ assures a denoising process that overcomes the undesired staircase effect [7].

The component $\varphi(\|\nabla K_\sigma * u\|)$ is introduced to avoid the blurring and enhance the image boundaries, thus assuring a detail-preserving additive noise removal. It is based on a positive function that is modeled as following:

$$\varphi : [0, \infty) \rightarrow [0, \infty),$$
$$\varphi(s) = \gamma \sqrt[3]{\delta s^2 + \nu} \quad (3)$$

where $\gamma, \delta, \nu \in [1, 3)$.

The argument of this function combines by convolution the evolving image with a two-dimension filter kernel that can be chosen as the 2D Gaussian kernel [1], having the form:

$$K_\sigma(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2 + y^2}{2\sigma^2}} \quad (4)$$

This combined anisotropic diffusion model is non-variational, since it cannot be derived from the minimization of an energy cost functional. It is also well-posed, admitting a unique weak solution representing the recovered image, which is computed by using the finite-difference numerical approximation scheme described in the next section.

III. FINITE DIFFERENCE-BASED NUMERICAL APPROXIMATION ALGORITHM

The proposed nonlinear fourth-order PDE-based model is then solved numerically by developing a finite difference method-based discretization algorithm [10]. For this purpose, we use a grid of space size h and the time step Δt and quantize the time and space coordinates, for a $[Ih \times Jh]$ support image, as follows:

$$x = ih, y = jh, t = n\Delta t,$$
$$i \in \{1, ..., I\}, j \in \{1, ..., J\}, n \in \{0, ..., N\} \quad (5)$$

The nonlinear fourth-order partial differential equation given by (1) leads to:

$$\frac{\partial u}{\partial t} + \lambda(u - u_0) = -\varphi(\|\nabla K_\sigma * u\|) \nabla^2 \left(\psi(\|\Delta u\|) \nabla^2 u \right) (6)$$

The left term of the equation (6) is discretized as following:

$$\frac{u_{i,j}^{n+\Delta t} - u_{i,j}^n}{\Delta t} + \lambda(u_{i,j}^n - u_{i,j}^0) = u_{i,j}^{n+\Delta t} \frac{1}{\Delta t} +$$
$$+ u_{i,j}^n \left(\lambda - \frac{1}{\Delta t} \right) - u_{i,j}^0 \lambda \quad (7)$$

which leads to

$$u_{i,j}^{n+1} + u_{i,j}^n (\lambda - 1) - u_{i,j}^0 \lambda \quad (8)$$

If we consider the values $h = \Delta t = 1$ for the time and step sizes.

Is right term is approximated next, using the finite difference-based Laplacian discretization [10] and the discretization $\varphi_{i,j} = \varphi\left(\left\| (K_\sigma * u)_{i,j} \right\| \right)$, where we have:

$$\|u_{i,j}\| \approx \sqrt{\left(\frac{u_{i+h,j} - u_{i-h,j}}{2h} \right)^2 + \left(\frac{u_{i,j+h} - u_{i,j-h}}{2h} \right)^2} \quad (9)$$

Therefore, we obtain the following approximation of $\varphi(\|\nabla K_\sigma * u\|) \nabla^2 \left(\psi(\|\Delta u\|) \nabla^2 u \right)$:

$$\varphi_{i,j} \Delta \psi_{i,j} = \varphi_{i,j} \frac{\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{i,j}}{h^2} \quad (10)$$

where we have

$$\psi_{i,j} = \psi\left(|\Delta u_{i,j}| \right) \nabla^2 u_{i,j} \quad (11)$$

and

$$\Delta u_{i,j} = \frac{u_{i+h,j}^n + u_{i-h,j}^n + u_{i,j+h}^n + u_{i,j-h}^n - 4u_{i,j}}{h^2} \quad (12)$$

Thus, by choosing $h = \Delta t = 1$, one obtains the following iterative explicit numerical approximation scheme:

$$u_{i,j}^{n+1} = u_{i,j}^n (1 - \lambda) + u_{i,j}^0 \lambda -$$
$$\varphi_{i,j} \left(\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} - 4\psi_{i,j} \right) \quad (13)$$

The iterative discretization algorithm (13) is consistent to the nonlinear fourth-order PDE model (1), converging to its solution in N iterations. This number of steps depends on the image size $[I \times J]$ and the amount of additive Gaussian noise.

This numerical approximation scheme has been successfully applied in our denoising tests which are described in the following section.

IV. RESTORATION EXPERIMENTS AND METHOD COMPARISON

We have performed a lot of restoration experiments, by applying the fourth-order diffusion-based denoising technique described here on hundreds of images affected by white additive Gaussian noise. The proposed PDE-based filtering approach removes successfully the additive noise and overcomes the undesirable effects, such as the blurring and staircasing, while preserving properly the boundaries, corners and other image features.

Method comparison have been also performed. The proposed method provides better restoration results than both the classic two-dimension filters, such as Gaussian and Average 2D filters, and many nonlinear PDE-based smoothing techniques, such as the Perona-Malik scheme and TV - ROF Denoising model.

The filtering performance of our restoration framework has been assessed using similarity measures such as Peak Signal to Noise Ratio (PSNR) and Mean-Shift Error (MSE) [11]. It achieves better values of these similarity metrics than the filtering schemes it has been compared with.

TABLE I.   METHOD COMPARISON: OBTAINED AVERAGE PSNRs

Restoration Method	Average PSNR value
Proposed technique	28.0134 (dB)
Average filter	25.4172 (dB)
Gaussian 2D filter	24.7655 (dB)
Perona-Malik scheme	26.9145 (dB)
TV-ROF Denoising	27.5226 (dB)
You-Kaveh scheme	27.8327 (dB)

One can see the average PSNR values obtained by our fourth-order nonlinear diffusion-based technique and other conventional and PDE-based models in Table I. The proposed approach achieves higher PSNR values than other existing methods.

A method comparison example is described in Fig. 1. The original *Boat* image is displayed in (a) and its version corrupted by an amount of white additive Gaussian noise given by the parameters: $\mu = 0.21$ and the variance is 0.02.
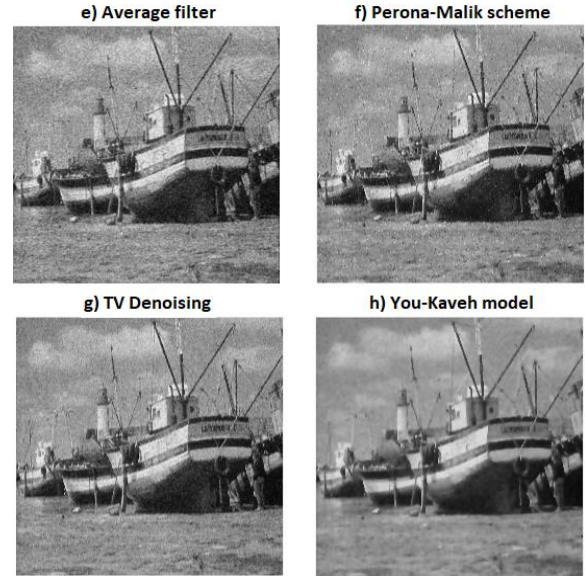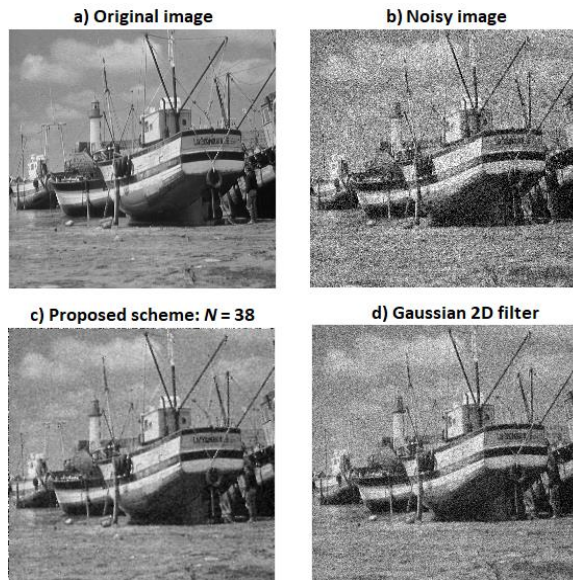

Figure 1.   Boat image filtered by several techniques

The restoration output obtained by our fourth-order diffusion-based technique after $N = 38$ iterations is displayed in (c). The image smoothed by conventional filters (Gaussian and Average 2D) is depicted in d) and e).

The restoration results achieved by second and fourth order PDE-based filters are displayed next: Perona-Malik scheme in (f), TV Denoising model in (g) and the You-Kaveh scheme in (h).
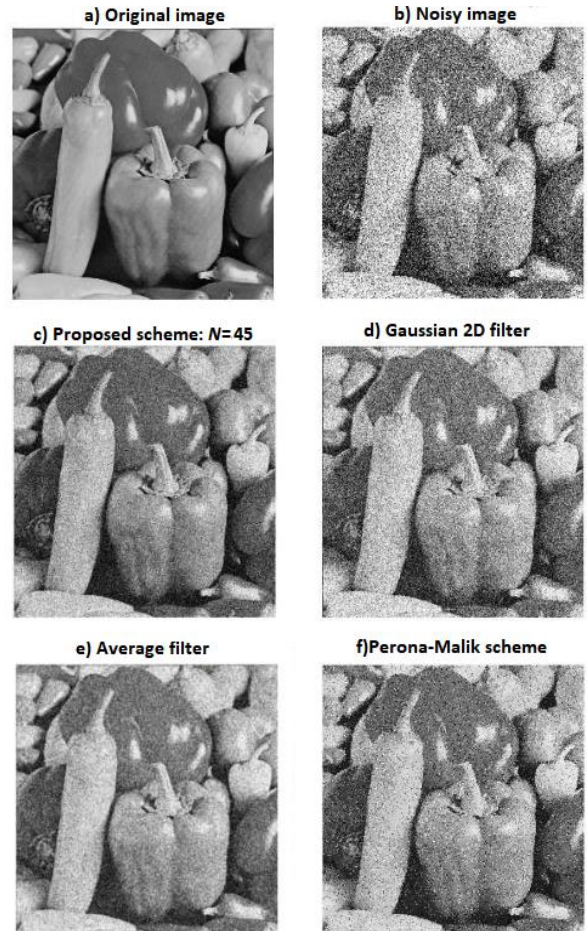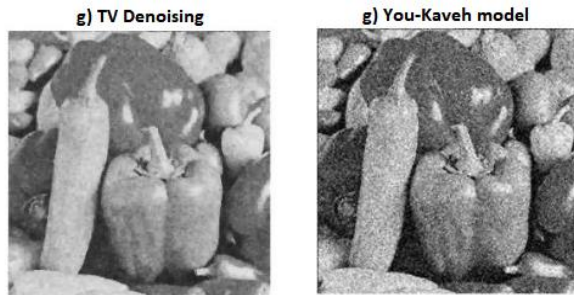
Figure 2. *Peppers* image denoised by various methods

Another method comparison example is displayed in the Fig. 2. So, the original *Peppers* image a) is affected by a higher amount of Gaussian noise in the image b).

Given the higher levels of the additive noise, the proposed PDE-based scheme needs more iterations to perform properly the image restoration and achieve the denoising output that is displayed in c) ($N = 45$ steps). The other filtering techniques, which are based on classic or PDE-based schemes, provide lower quality image denoising results which are depicted in the images from d) to f).

V. CONCLUSIONS

A nonlinear fourth-order diffusion-based image restoration framework has been proposed in this research article. It is based on a novel fourth-order PDE-based model, which represents the main contribution of this paper.

The differential model developed here is much more complex than other fourth-order PDE denoising schemes derived from You-Kaveh algorithm. It combines the fourth-order diffusion component to another term, which is based on the convolution of the evolving image to a conventional 2D filter kernel, and enhances the edges and other details.

The finite difference-based numerical approximation scheme that solves numerically this nonlinear diffusion model is another contribution of this research paper. It is stable, consistent to the described PDE model, converges fast to its solution and provides very good restoration results when applied on images corrupted by additive Gaussian noise.

Our restoration approach produces an effective edge-preserving image denoising and overcomes the most common unintended effects. It also outperforms not only the classic image filters, but also numerous state of the art second and fourth-order PDE-based smoothing techniques.

The filtering method proposed here can be successfully be applied in the PDE-based inpainting domain. Therefore, developing a novel effective fourth-order diffusion-based image interpolation scheme that is derived from the PDE denoising model described in this work will represent a focus of our future research in this field.

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